JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **217,** 335–348 (1998)
ARTICLE NO. AY975725

Controllability of Sobolev-Type Integrodifferential Systems in Banach Spaces

K. Balachandran

Department of Mathematics, Bharathiar University, Coimbatore, Tamil Nadu, India

and

J. P. Dauer

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, Tennessee

Submitted by S. M. Meerkov

Received July 8, 1996

Sufficient conditions for controllability of Sobolev-type integrodifferential systems in Banach spaces are established. The results are obtained using compact semigroups and the Schauder fixed-point theorem. As an example is provided to illustrate the results. © 1998 Academic Press

Key Words: Controllability; integrodifferential systems; Sobolev-type systems; Schauder's fixed-point theorem.

1. INTRODUCTION

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional spaces has been extensively studied. Several authors [5, 6, 14–18, 22, 25] have extended the concept to infinite dimensional systems in Banach spaces with bounded operators. Triggiani [23] established sufficient conditions for controllability of linear and nonlinear systems in Banach spaces. Exact controllability of abstract semilinear equations has been studied by Lasiecka and Triggiani [12]. Quinn and Carmichael [21] have shown that the controllability problem in Banach spaces can be converted into one of a fixed-point problem for a single-valued mapping. Kwun *et al.* [11] investigated the controllabil-

0022-247X/98 $25.00

ity and approximate controllability of delay Volterra systems by using a fixed-point theorem. Recently Balachandran *et al*. [1, 3] studied the controllability and local null controllability of nonlinear integrodifferential systems and functional differential systems in Banach spaces. The purpose of this paper is to study the controllability of Sobolev-type integrodifferential systems in Banach spaces by using the Schauder fixed-point theorem. The integrodifferential equation considered here serves as an abstract formulation of Sobolev-type integrodifferential equations which arise in many physical phenomena [1, 7–9, 13, 20].

Consider a nonlinear integrodifferential system of the form

$$\begin{aligned}(Ex(t))' + Ax(t) &= Bu(t) + f(t, x(t)) \\ &\quad + \int_0^t g\left(t, s, x(s), \int_0^s H(s, \tau, x(\tau))\, d\tau\right) ds, \\ &\qquad t \in J = [0, a],\ a > 0, \\ x(0) &= x_0 \qquad (1)\end{aligned}$$

where the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J, U)$, the Banach space of admissible control functions with U a Banach space. B is a bounded linear operator from U into Y, a Banach space. The nonlinear operators $f \in C(J \times X, Y)$, $H \in C(J \times J \times X, X)$, and $g \in C(J \times J \times X \times X, Y)$ are all uniformly bounded continuous operators.

For such a Sobolev-type equation, Brill [4] has studied the abstract Cauchy problem for semilinear evolution equation when $B = 0$ and $g = 0$. Also, Kartsatos and Parrott [10] have studied pseudoparabolic problems with operators $A(t, u_t)$, $B = 0$ and $g = 0$. In particular, this paper is motivated by recent work on controllability by Balachandran *et al*. [1–3] and by Park and Han [9].

2. PRELIMINARIES

The operators A: $D(A) \subset X \to Y$ and E: $D(E) \subset X \to Y$ satisfy the hypotheses $[C_i]$ for $i = 1, 2, \ldots, 4$:

$[C_1]$ A and E are closed linear operators,

$[C_2]$ $D(E) \subset D(A)$ and E is bijective,

$[C_3]$ E^{-1}: $Y \to D(E)$ is continuous.

$[C_4]$ For each $t \in [0, a]$ and for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, the resolvent $R(\lambda, -AE^{-1})$ is a compact operator.

The hypotheses $[C_1]$, $[C_2]$ and the closed graph theorem imply the boundedness of the linear operator $AE^{-1}: Y \to Y$.

LEMMA [20]. *Let $S(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda: A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

From the above fact, $-AE^{-1}$ generates a compact semigroup $T(t), t \geq 0$. Thus, $\max_{t \in J}\|T(t)\|$ is finite and so denote $M = \max_{t \in J}\|T(t)\|$.

DEFINITION. The system (1) is said to be controllable on the interval J if for every $x_0, x_1 \in X$ there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(a) = x_1$.

$[C_5]$ The linear operator W from U into X is defined by

$$Wu = \int_0^a E^{-1}T(a - s)Bu(s)\,ds,$$

where there exists a bounded invertible operator W^{-1} defined on $L^2(J, U)/\ker W$ and B is a bounded linear operator.

$[C_6]$ The function f satisfies the following two conditions:

(i) For each $t \in J$, the function $f(t, \cdot): X \to Y$ is continuous, and for each $x \in X$ the function $f(\cdot, x): J \to Y$ is strongly measurable.

(ii) For each natural number k, there is a function $g_k \in L^1(J)$ such that

$$\sup_{|x| \leq k} |f(t, s)| \leq g_k(t),$$

$$\lim_{k \to \infty} \inf \frac{1}{k}\int_0^a g_k(s)\,ds = \alpha < \infty,$$

where α is a real number.

$[C_7]$ For each $(t, s) \in J \times J$ the function $H(t, s, \cdot): X \to X$ is continuous, and for each $x \in X$ the function $H(\cdot, \cdot, x): J \times J \to X$ is strongly measurable.

$[C_8]$ The function g satisfies the following two conditions:

(i) For each $(t, s, x) \in J \times J \times X$ the function $g(t, s, \cdot, \cdot): X \times X \to Y$ is continuous, and for each $x \in X$, $H \in X$ the function $g(\cdot, x, y): J \times J \to Y$ is strongly measurable.

(ii) For each number k, there is a function $h_k \in L^1(J)$ such that

$$\sup_{|x|<k} \left| \int_0^t g\left(t, s, x, \int_0^s H(s, \tau, x)\, d\tau\right) ds \right| \leq h_k(t),$$

$$\lim_{k\to\infty} \inf \frac{1}{k} \int_0^a h_k(t)\, dt = \beta < \infty,$$

where β is a real number.

Now the solution of (1) is given by the integral equation

$$\begin{aligned} x(t) = E^{-1}T(t)Ex_0 &+ \int_0^t E^{-1}T(t-s)f(s, x(s))\, ds \\ &+ \int_0^t E^{-1}T(t-s)Bu(s)\, ds \\ &+ \int_0^t E^{-1}T(t-s) \int_0^s g\left(s, \tau, x(\tau), \int_0^\tau H(\tau, \eta, x(\eta))\, d\eta\right) d\tau ds. \end{aligned} \tag{2}$$

Let $Q(t) = \int_0^t H(t, s, x(s))\, ds$. Then (2) can be written as

$$\begin{aligned} x(t) = E^{-1}T(t)Ex_0 &+ \int_0^t E^{-1}T(t-s)f(s, x(s))\, ds \\ &+ \int_0^t E^{-1}T(t-s)Bu(s)\, ds \\ &+ \int_0^t E^{-1}T(t-s) \int_0^s g(s, \tau, x(\tau), Q(\tau))\, d\tau ds. \end{aligned}$$

In the next section the Schauder fixed-point theorem is used to establish the controllability theorem for Eq. (1) under the above conditions.

3. MAIN RESULT

THEOREM. *If the assumptions* $[C_1]$–$[C_8]$ *are satisfied, then the system* (1) *is controllable on J provided that*

$$(\alpha + \beta) M \|E^{-1}\| [1 + aM\|B\|\, \|W^{-1}\|\, \|E^{-1}\|] < 1.$$

Proof. Using the assumption $[C_5]$, for an arbitrary function $x(\cdot)$ define the control

$$\begin{aligned} u(t) = W^{-1}\Bigg[x_1 - E^{-1}T(a)Ex_0 &- \int_0^a E^{-1}T(a-s)f(s, x(s))\, ds \\ &- \int_0^a E^{-1}T(a-s)\left(\int_0^s g(s, \tau, x(\tau), Q(\tau))\, d\tau\right) ds \Bigg](t). \end{aligned}$$

It shall now be shown that when using this control, the operator S defined by

$$
\begin{aligned}
(Sx)(t) = {} & E^{-1}T(t)Ex_0 + \int_0^t E^{-1}T(t-s)f(s, x(s))\,ds \\
& + \int_0^t E^{-1}T(t-s)Bu(s)\,ds \\
& + \int_0^t E^{-1}T(t-s)\left\{\int_0^s g(s, \tau, x(\tau), Q(\tau))\,d\tau\right\}ds
\end{aligned}
$$

from $C(J, X)$ into itself for each $x \in C = C(J, X)$ has a fixed point. This fixed point is then a solution of Eq. (1).

Clearly,

$$
\begin{aligned}
(Sx)(a) = {} & E^{-1}T(a)Ex_0 + \int_0^a E^{-1}T(a-s)f(s, x(s))\,ds \\
& + \int_0^a E^{-1}T(a-s)Bu(s)\,ds \\
& + \int_0^a E^{-1}T(a-s)\left\{\int_0^s g(s, \tau, x(\tau), Q(\tau))\,d\tau\right\}ds \\
= {} & E^{-1}T(a)Ex_0 + \int_0^a E^{-1}T(a-\tau)f(\tau, x(\tau))\,ds \\
& + \int_0^a E^{-1}T(a-s)B\Bigg[W^{-1}\Big[x_1 - E^{-1}T(a)Ex_0 \\
& \qquad - \int_0^a E^{-1}T(a-\tau)f(\tau, x(\tau))\,d\tau \\
& - \int_0^a E^{-1}T(a-\tau)\left\{\int_0^\tau g(\tau, \eta, x(\eta), Q(\eta)\,d\eta\right\}d\tau\Big](s)\Bigg]\,ds \\
& + \int_0^a E^{-1}T(a-s)\left\{\int_0^s g(s, \tau, x(\tau), Q(\tau)\,d\tau\right\}\,ds \\
= {} & x_1
\end{aligned}
$$

It can be easily verified that S maps C into itself continuously. For each natural number k let

$$B_k = \{x \in C\colon x(0) = x_0, \|x(t)\| \le k, t \in J\}.$$

Then for each k, the set B_k is clearly a bounded, closed, convex subset in C and there exists a natural number K with $SB_K \subset B_K$. If this is not the case, then for each natural number k there is a function $x_k \in B_k$ with $Sx_k \notin B_k$, that is,

$$\|Sx_k\| \geq k.$$

Then $1 \leq \frac{1}{k}\|Sx_k\|$, and so

$$1 \leq \varliminf_{k\to\infty} k^{-1}\|Sx_k\|.$$

However,

$$\begin{aligned}
&\varliminf_{k\to\infty} k^{-1}\|Sx_k\| \\
&\quad\leq \varliminf_{k\to\infty} k^{-1}\Big\{M\|E^{-1}\|\,\|E\|\,\|x_0\| + M\|E^{-1}\|\int_0^a g_k(s)\,ds \\
&\qquad\qquad + M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|\int_0^a \Big[\|x_1\| + \|E^{-1}\|M\|E\|\,\|x_0\| \\
&\qquad\qquad\qquad + \|E^{-1}\|M\int_0^a g_k(\tau)\,d\tau + \|E^{-1}\|M\int_0^a h_k(\tau)\,d\tau\Big]ds \\
&\qquad\qquad + \|E^{-1}\|M\int_0^a h_k(s)\,ds\Big\} \\
&\quad= \alpha M\|E^{-1}\| + \alpha a M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|\,\|E^{-1}\|M \\
&\qquad + \beta a M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|\,\|E^{-1}\|M + \beta\|E^{-1}\|M \\
&\quad= \alpha M\|E^{-1}\|\big[1 + aM\|B\|\,\|W^{-1}\|\,\|E^{-1}\|\big] \\
&\qquad + \beta M\|E^{-1}\|\big[1 + aM\|B\|\,\|W^{-1}\|\,\|E^{-1}\|\big] \\
&\quad= (\alpha + \beta)M\|E^{-1}\|\big[1 + aM\|B\|\,\|W^{-1}\|\,\|E^{-1}\|\big] < 1,
\end{aligned}$$

a contradiction. Hence, $SB_K \subset B_K$ for some positive integer K.

In fact, the operator S maps B_K into a compact subset of B_K. To prove this, it is first shown that for every fixed $t \in J$ the set

$$V_K(t) = \{(Sx)(t) : x \in B_K\}$$

is a precompact in X. This is trivial for $t = 0$, since $V_K(0) = \{x_0\}$. So let t, $0 < t \leq a$, be fixed and let ϵ be a given real number satisfying $0 < \epsilon < t$.

Define

$$
\begin{aligned}
(S_\epsilon x)(t) &= E^{-1}T(t)Ex_0 + \int_0^{t-\epsilon} E^{-1}T(t-s)f(s,x(s))\,ds \\
&\quad + \int_0^{t-\epsilon} E^{-1}T(t-s)Bu(s)\,ds \\
&\quad + \int_0^{t-\epsilon} E^{-1}T(t-s)\left\{\int_0^s g(s,\tau,x(\tau),Q(\tau))\,d\tau\right\}ds \\
&= E^{-1}T(t)Ex_0 + \int_0^{t-\epsilon} E^{-1}T(t-s)f(s,x(s))\,ds \\
&\quad + \int_0^{t-\epsilon} E^{-1}(t-s)BW^{-1}\Bigg[x_1 - E^{-1}T(a)Ex_0 \\
&\qquad - \int_0^a E^{-1}T(a-\tau)f(\tau,x(\tau))\,d\tau \\
&\qquad - \int_0^a E^{-1}T(a-\tau)\int_0^\tau g(\tau,\eta,x(\eta),Q(\eta)\,d\eta)\,d\tau\Bigg](\lambda)\,ds \\
&\quad + \int_0^{t-\epsilon} E^{-1}T(t-s)\left\{\int_0^s g(s,\tau,x(\tau),Q(\tau))\,d\tau ds.\right.
\end{aligned}
$$

Since $u(s)$ is bounded and $T(t)$ is compact, the set $V_\epsilon(t) = \{(S_\epsilon x)(t)\colon x \in B_K\}$ is a precompact set in X. Also, for $x \in B_K$, using the defined control $u(t)$ yields

$$
\begin{aligned}
&\|(Sx)(t) - (S_\epsilon x)(t)\| \\
&\quad = \|E^{-1}T(t)Ex_0 + \int_0^t E^{-1}T(t-s)f(s,x(s))\,ds \\
&\qquad + \int_0^t E^{-1}T(t-s)Bu(s)\,ds \\
&\qquad + \int_0^t E^{-1}T(t-s)\left\{\int_0^s g(s,\tau,x(\tau),Q(\tau))\,d\tau\right\}ds \\
&\qquad - E^{-1}T(t)Ex_0 - \int_0^{t-\epsilon} E^{-1}T(t-s)f(s,x(s))\,ds \\
&\qquad - \int_0^{t-\epsilon} E^{-1}T(t-s)Bu(s)\,ds \\
&\qquad - \int_0^{t-\epsilon} E^{-1}T(t-s)\left\{\int_0^s g(s,\tau,x(\tau),Q(\tau))\,d\tau\right\}ds\|
\end{aligned}
$$

$$\leq \left\| \int_{t-\epsilon}^{t} E^{-1}T(t-s)f(s,x(s))\,ds \right\| + \left\| \int_{t-\epsilon}^{t} E^{-1}T(t-s)Bu(s)\,ds \right\|$$

$$+ \left\| \int_{t-\epsilon}^{t} E^{-1}T(t-s)\left\{ \int_0^s g(s,\tau,x(\tau),Q(\tau))\,d\tau \right\} ds \right\|$$

$$\leq M\|E^{-1}\| \int_{t-\epsilon}^{t} |f(s,x(s))| ds + M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|$$

$$\times \int_{t-\epsilon}^{t} \Bigg[\|x_1\| + M\|E^{-1}\|\,\|E\|\,\|x_0\| + M\|E^{-1}\| \int_0^a |f(\tau,x(\tau))|\,d\tau$$

$$+ M\|E^{-1}\| \int_0^a \left\{ \int_0^t |g(\tau,\eta,x(\eta),Q(\eta)\,d\eta| d\tau \right\}(s)\,ds$$

$$+ M\|E^{-1}\| \int_{t-\epsilon} \left\{ \int_0^s |g(s,\tau,x(\tau),Q(\tau))\,d\tau| \right\} ds$$

$$\leq M\|E^{-1}\| \int_{t-\epsilon}^{t} g_K(s)\,ds + M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|$$

$$\times \int_{t-\epsilon}^{t} \Bigg[\|x_1\| + M\|E^{-1}\|\,\|E\|\,\|x_0\| + M\|E^{-1}\| \int_0^a g_K(\tau)\,d\tau$$

$$+ M\|E^{-1}\| \int_0^a h_K(\tau)\,d\tau \Bigg] ds + M\|E^{-1}\| \int_{t-\epsilon}^{t} h_K(s)\,ds.$$

Since $g_K, h_K \in L^1(J)$, it follows that $\|(Sx)(t) - (S_\epsilon x)(t)\|$ is finite by the uniform boundedness principle. Thus, there are precompact sets arbitrarily close to the set $V_K(t)$ and so $V_K(t)$ is precompact in X.

Next it is shown that

$$SB_K = \{Sx : x \in B_K\}$$

is an equicontinuous family of functions. Let $x \in B_K$ and $t, \tau \in J$ such that $0 < t < \tau$; then

$$\|(Sx)(t) - (Sx)(\tau)\|$$

$$\leq \|T(t) - T(\tau)\|\,\|E^{-1}\|E\|x_0\|$$

$$+ \int_0^t \|T(t-s) - T(\tau-s)\|\,\|E^{-1}\|\,|f(s,x(s))|ds$$

$$+\int_t^\tau \|T(\tau-s)\|\,\|E^{-1}\|\,|f(s,x(s))|ds$$

$$+\int_0^t \|T(t-s)-T(\tau-s)\|\,\|E^{-1}\|\,\|B\|\,\|W^{-1}\|$$

$$\times\Bigg[\|x_1\|+\|E^{-1}\|\,\|T(a)\|\,\|E\|\,\|x_0\|$$

$$+\int_0^a \|E^{-1}\|\,\|T(a-\tau)\|\,|f(\tau,x(\tau))|d\tau$$

$$+\int_0^a \|E^{-1}\|\,\|T(a-\tau)\|\left\{\int_0^s |g(s,\eta,x(\eta),Q(\eta))\,d\eta|\right\}d\tau\Bigg](s)\,ds$$

$$+\int_t^\tau \|T(\tau-s)\|\,\|E^{-1}\|\,\|B\|\,\|W^{-1}\|$$

$$\times\Bigg[\|x_1\|+\|E^{-1}\|\,\|T(a)\|\,\|E\|\,\|x_0\|$$

$$+\int_0^a \|E^{-1}\|\,\|T(a-\tau)\|\,|f(\tau,x(\tau))|\,d\tau$$

$$+\int_0^a \|E^{-1}\|\,\|T(a-\tau)\|\left\{\int_0^s |g(s,\eta,x(\eta),Q(\eta))\,d\eta|\right\}d\tau\Bigg](s)\,ds$$

$$+\int_0^t \|T(t-s)-T(\tau-s)\|\,\|E^{-1}\|\left(\int_0^s |g(s,\eta,x(\eta),Q(\eta))\,d\eta|\right)ds$$

$$+\int_t^\tau \|T(\tau-s)\|\,\|E^{-1}\|\left\{\int_0^s |g(s,\eta,x(\eta),Q(\eta))\,d\eta|\right\}ds$$

$$\leq \|T(t)-T(\tau)\|\,\|E^{-1}\|\,\|E\|\,\|x_0\|$$

$$+\int_0^t \|T(t-s)-T(\tau-s)\|\,\|E^{-1}\|g_K(s)\,ds$$

$$+\int_t^\tau \|T(t-s)\|\,\|E^{-1}\|g_K(s)\,ds$$

$$+\int_0^t \|T(t-s)-T(\tau-s)\|\,\|E^{-1}\|\,\|B\|\,\|W^{-1}\|$$

$$\times\Bigg[\|x_1\|+\|E^{-1}\|\,\|T(a)\|\,\|E\|\,\|x_0\|+\int_0^a \|E^{-1}\|\,\|T(a-\tau)\|g_K(\tau)\,d\tau$$

$$+\int_0^a \|E^{-1}\|\,\|T(a-\tau)\|h_K(\tau)\,d\tau\Bigg](s)\,ds$$

$$
\begin{aligned}
&+\int_t^\tau \|T(\tau-s)\|\,\|E^{-1}\|\,\|B\|\,\|W^{-1}\| \\
&\times\Bigg[\|x_1\| + \|E^{-1}\|\,\|T(a)\|\,\|E\|\,\|x_0\| + \int_0^a \|E^{-1}\|\,\|T(a-\tau)\|g_K(\tau) \\
&\qquad\qquad + \int_0^a \|E^{-1}\|\,\|T(a-\tau)\|h_K(\tau)\,d\tau\Bigg]ds \\
&+\int_0^t \|T(t-s) - T(\tau-s)\|\,\|E^{-1}\|h_K(s)\,ds \\
&+\int_t^\tau \|T(\tau-s)\|\,\|E^{-1}\|h_K(s)\,ds
\end{aligned}
$$

Now $T(t)$ is continuous in the uniform operator topology for $t > 0$. Since $T(t)$ is compact and $g_K, h_K \in L^1(J)$, the right hand side of above inequality tends to zero as $t \to \tau$. Thus, SB_K is equicontinuous and also bounded. By the Arzela–Ascoli theorem SB_K is precompact in $C(J, X)$. Hence S is a completely continuous operator on $C(J, X)$. From the Schauder fixed-point theorem, S has a fixed point in B_K. Any fixed point of S is a mild solution of (1) on J satisfying $(Sx)(t) = x(t) \in X$. Thus, the system (1) is controllable on J. ∎

4. EXAMPLE

The result from Section 3 is illustrated by showing its applicability to a partial integrodifferential equation with nonlinear functions satisfying the Caratheodory condition.

Consider the following differential equation with control term

$$
\begin{aligned}
&\frac{\partial}{\partial t}\big(z(t,x) - z_{xx}(t,x)\big) - z_{xx}(t,x) \\
&\qquad = Bu(t) + \mu_1\big(t, z_{xx}(t,x)\big) \\
&\qquad\qquad + \int_0^t \mu_3\left(t, s, z_{xx}(s,x), \int_0^s \mu_2\big(s,\tau,z_{xx}(\tau,x)\big)d\tau\right)ds
\end{aligned}
$$

$$
\text{where } x \in [0,\pi] \text{ and } t \in J \tag{3}
$$

$$
\begin{aligned}
z(t,0) &= z(t,\pi) = 0, && t \in J \\
z(0,x) &= z_0(x), && x \in [0,\pi].
\end{aligned}
$$

It is assumed that the following conditions hold with $X = Y = L^2[0,\pi]$.

$[A_1]$ The operator $B\colon U \to Y$, with $U \subset J$, is a bounded linear operator.

$[A_2]$ The linear operator $W\colon U \to X$ is defined by

$$Wu = \int_0^a E^{-1}T(a-s)Bu(s)\,ds$$

and has a bounded invertible operator W^{-1} defined on $L^2(J,U)/\ker W$.

$[A_3]$ The nonlinear operator $\mu_1\colon J \times X \to Y$ satisfies the following three conditions:

(i) For each $t \in J$, $\mu_1(t,z)$ is continuous.

(ii) For each $z \in X$, $\mu_1(t,z)$ is measurable.

(iii) There is a constant $\nu (0 < \nu < 1)$ and a function $h \in L^1(J)$ such that for all $(t,z) \in J \times X$,

$$\|\mu_1(t,z)\| \le h(t)|z|^\nu.$$

$[A_4]$ The nonlinear operator $\mu_2\colon J \times J \times X \to X$ satisfies the following two conditons:

(i) For each $(t,s) \in J \times J$, $\mu_2(t,s,z)$ is continuous.

(ii) For each $z \in X$, $\mu_2(t,s,z)$ is measurable.

$[A_5]$ The nonlinear operator $\mu_3\colon J \times J \times X \times X \to Y$ satisfies the following three conditions:

(i) For each $(t,s,z) \in J \times J \times X$, $\mu_3(t,s,z)$ is continuous.

(ii) For each $z \in X$, $\mu_3(t,s,z)$ is measurable.

(iii) There is a constant $\nu (0 < \nu < 1)$ and a function $k \in L^1(J)$ such that

$$\left|\int_0^t \mu_3\left(t,s,z,\int_0^s \mu_2(s,\tau,z)\,d\tau\right)ds\right| \le k(t)|z|^\nu$$

for all $(t,s,z,y) \in J \times J \times X \times X$.

Define the operators $A\colon D(A) \subset X \to Y$, $E\colon D(E) \subset X \to Y$ by

$$Az = -z_{xx},$$

$$Ez = z - z_{xx},$$

respectively, where each domain $D(A)$, $D(E)$ is given by

$$\{z \in X\colon z, z_x \text{ are absolutely continuous, } z_{xx} \in X,\ z(0) = z(\pi) = 0\}.$$

Define an operator $f\colon J \times X \to Y$ by

$$f(t,z)(x) = \mu_1(t, z_{xx}(x))$$

and let

$$H(t,s,z)(x) = \mu_2(t,s,z_{xx}(x)), \qquad (t,s,z) \in J \times J \times X,$$

$$g\left(t,s,z,\int_0^s H(t,s,z)\right)(x) = \mu_3\left(t,s,z_{xx}(x),\int_0^s \mu_2(t,s,z_{xx}(x))\,dt\right),$$
$$x \in [0,\pi].$$

Then the above problem (3) can be formulated abstractly as

$$(Ez(t))' + Az(t) = Bu(t) + f(t,z(t))$$
$$+ \int_0^t g\left(t,s,z(s),\int_0^s h(s,\tau,z(\tau))\,d\tau\right) ds, \qquad t \in J,$$
$$z(0) = z_0.$$

Also, A and E can be written, respectively, as (see [13])

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, z_n\rangle z_n, \qquad z \in D(A),$$

$$Ez = \sum_{n=1}^{\infty} (1+n^2)\langle z, z_n\rangle z_n, \qquad z \in D(E),$$

where $z_n(x) = \sqrt{2/\pi}\,\sin nx$, $n = 1, 2, \ldots,$ is the orthonormal set of eigenfunctions of A. Furthermore, for $z \in X$ we have

$$E^{-1}z = \sum_{n=1}^{\infty} \frac{1}{(1+n^2)} \langle z, z_n\rangle z_n,$$

$$-AE^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2)} \langle z, z_n\rangle z_n,$$

$$T(t)z = \sum_{n=1}^{\infty} e^{\frac{-n^2}{(1+n^2)}t} \langle z, z_n\rangle z_n.$$

It is easy to see that $-AE^{-1}$ generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $\|T(t)\| \le e^{-t}$ for each $t > 0$. Also, the operator f satisfies condition $[C_6]$ [24] and the operators g and K satisfy $[C_7]$ and $[C_8]$. So all the conditions stated in the above theorem are satisfied. Hence the system (3) is controllable on J.

ACKNOWLEDGMENTS

The first author is thankful to the Fulbright Foundation for providing a travel grant to visit the University of Tennessee at Chattanooga and to Professor J. P. Dauer for providing a hospitable stay while at UTC during June and July 1996.

REFERENCES

1. K. Balachandran, P. Balasubramaniam, and J. P. Dauer, Controllability of nonlinear integrodifferential systems in Banach space, *J. Optim. Theory Appl.* **84** (1995), 83–91.
2. K. Balachandran, P. Balasubramaniam, and J. P. Dauer, Controllability of quasilinear delay systems in Banach spaces, *Optimal Control Appl. Methods* **16** (1995), 283–290.
3. K. Balachandran, P. Balasubramaniam, and J. P. Dauer, Local null controllability of nonlinear functional differential systems in Banach spaces, *J. Optim. Theory Appl.* **88** (1996), 61–75.
4. H. Brill, A semilinear Sobolev equation in Banach space, *J. Differential Equations* **24** (1977), 412–425.
5. E. N. Chuckwu and S. M. Lenhart, Controllability questions for nonlinear systems in abstract spaces, *J. Optim. Theory Appl.* **68** (1991), 437–462.
6. R. F. Curtain and A. J. Prichard, "Infinite Dimensional Linear Systems Theory," Springer-Verlag, New York, 1978.
7. W. E. Fitzgibbon, Semilinear integrodifferential equations in a Banach space, *Nonlinear Analysis Theory Methods Appl.* **4** (1980), 745–760.
8. M. L. Heard, An abstract semilinear hyperbolic Volterra integrodifferential equation, *J. Math. Anal. Appl.* **80** (1981), 175–202.
9. M. A. Hussain, On a nonlinear integrodifferential equation in Banach space, *Indian J. Pure Appl. Math.* **19** (1988), 516–529.
10. A. G. Kartsatos and M. E. Parrott, On a class of nonlinear functional pseudoparabolic problems, *Funkcial. Ekvac.* **25** (1982), 207–221.
11. Y. C. Kwun, J. Y. Park, and J. W. Ryu, Approximate controllability and controllability for delay Volterra systems, *Bull. Korean Math. Soc.* **28** (1991), 131–145.
12. I. Lasiecka and R. Triggiani, Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems, *Appl. Math. Opt.* **23** (1991), 109–154.
13. J. H. Lightbourne III and S. M. Rankin III, A partial functional differential equation of Sobolev type, *J. Math. Anal. Appl.* **93** (1983), 328–337.
14. S. Nakagiri and R. Yamamoto, Controllability and observability of linear retarded systems in Banach spaces, *Internat. J. Control* **49** (1989), 1489–1504.
15. K. Naito, Controllability of semilinear control systems dominated by the linear part, *SIAM J. Control Optim.* **25** (1987), 715–722.
16. K. Naito, Approximate controllability for trajectories of semilinear control systems, *J. Optim. Theory Appl.* **6** (1989), 57–65.
17. K. Naito, On controllability for a nonlinear Volterra equation, *Nonlinear Anal. Theory Methods Appl.* **18** (1992), 99–108.
18. K. Naito and J. Y. Park, Approximate controllability for trajectories of a delay Volterra control systems, *J. Optim. Theory Appl.* **61** (1989), 271–279.
19. J. Y. Park and H. K. Han, Controllability for Sobolev type differential equations, to appear.
20. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.

21. M. D. Quinn and N. Carmichael, An approach to nonlinear control problems using fixed-point methods, degree theory, and pseudo-inverses, *Numer. Funct. Anal. Optim.* **7** (1984–1985), 197–219.
22. D. L. Russell, Controllability and stabilizability theory for linear partial differential equations; Recent progress and open questions, *SIAM Rev.* **20** (1978), 639–673.
23. R. Triggiani, Controllability and observability in Banach space with bounded operators, *SIAM J. Control* **13** (1975), 462–491.
24. J. R. Ward, Boundary value problems for differential equations in Banach spaces, *J. Math. Anal. Appl.* **70** (1979), 589–598.
25. H. X. Zhou, Approximate controllability for a class of semilinear abstract equations, *SIAM J. Control Optim.* **21** (1983), 551–565.